

# THE CONTINUITY OF SEQUENTIAL PRODUCT OF SEQUENTIAL QUANTUM EFFECT ALGEBRAS

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**ABSTRACT.** In order to study quantum measurement theory, sequential product defined by  $A \circ B = A^{1/2}BA^{1/2}$  for any two quantum effects  $A, B$  is introduced. Physically motivated conditions ask the sequential product to be continuous with respect to the strong operator topology. In this paper, we study the continuity problems of the sequential product  $A \circ B = A^{1/2}BA^{1/2}$  with respect to the other important topologies, as norm topology, weak operator topology, order topology, interval topology, etc.

## 1. INTRODUCTION

Effect algebra is an important model for studying the unsharp quantum logic, it were introduced by D. J. Foulis and M. K. Bennett in 1994, that is

**Definition 1.1.** ([1]). A structure  $(E; \oplus, 0, 1)$  is called an effect algebra if  $0, 1$  are two distinguished elements and  $\oplus$  is a partially defined binary operation on  $E$  which satisfies the following conditions for any  $a, b, c \in E$ :

- (E1) If  $a \oplus b$  is defined, then  $b \oplus a$  is defined and  $a \oplus b = b \oplus a$ .
- (E2) If  $a \oplus b$  and  $(a \oplus b) \oplus c$  are defined, then  $b \oplus c$  and  $a \oplus (b \oplus c)$  are defined and  $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ .
- (E3) For each  $a \in E$ , there exists a unique  $b \in E$  such that  $a \oplus b$  is defined and  $a \oplus b = 1$ .
- (E4) If  $a \oplus 1$  is defined, then  $a = 0$ .

In an effect algebra  $(E, 0, 1, \oplus)$ , if  $a \oplus b$  is defined, we write  $a \perp b$ . For each  $a \in (E, 0, 1, \oplus)$ , it follows from (E3) that there exists a unique element  $b \in E$  such that  $a \oplus b = 1$ , we denote  $b$  by  $a'$ . Let  $a, b \in (E, 0, 1, \oplus)$ , if there exists a  $c \in E$  such that  $a \perp c$  and  $a \oplus c = b$ , then we say that  $a \leq b$  and define  $c = b \ominus a$ . Thus, each effect algebra  $(E, 0, 1, \oplus)$  has two partially defined binary operations  $\oplus$  and  $\ominus$ . Moreover, it follows from ([1]) that  $\leq$  is a partial order of  $(E, 0, 1, \oplus)$  and satisfies that for each  $a \in E$ ,  $0 \leq a \leq 1$ ,  $a \perp b$  if and only if  $a \leq b'$ .

The most important and prototype of effect algebras is  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus)$ , where  $\mathcal{H}$  is a complex Hilbert space,  $\mathcal{E}(\mathcal{H})$  is the set of all quantum effects, that is, all positive operators on  $\mathcal{H}$  that are bounded above by the identity operator  $I$ , the partial binary operation  $\oplus$  is defined for  $A, B \in \mathcal{E}(\mathcal{H})$  iff  $A + B \leq I$ , in this case,  $A \oplus B = A + B$ .

One can use quantum effects to represent the yes-no measurements that may be unsharp ([1]).

Let  $\mathcal{D}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$  be the set of density operators on  $\mathcal{H}$ , that is, the trace class positive operators on  $\mathcal{H}$  of unit trace, and  $\mathcal{P}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H})$  the set of orthogonal

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*Key words and phrases.* Quantum effects, Sequential product, Continuity, Topology.

projections on  $\mathcal{H}$ . For each  $P \in \mathcal{P}(\mathcal{H})$ , there is associated a so-called Lüders transformation  $\Phi_L^P : \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H})$  such that for each  $T \in \mathcal{D}(\mathcal{H})$ ,  $\Phi_L^P(T) = PTP$ . Moreover, each quantum effect  $B \in \mathcal{E}(\mathcal{H})$  gives also a general Lüders transformation  $\Phi_L^B$  such that  $\Phi_L^B(T) = B^{\frac{1}{2}}TB^{\frac{1}{2}}$  ([2, 3]).

For  $A, B \in \mathcal{E}(\mathcal{H})$ ,  $A^{1/2}BA^{1/2}$  is called the sequential product of  $A$  and  $B$  by Gudder and denoted by  $A \circ B$  ([4, 5, 6]). The product  $A \circ B$  represents the effect produced by first measuring  $A$  then measuring  $B$ . This product has also been generalized to an algebraic structure called a sequential effect algebra ([7]), that is

**Definition 1.2.** ([7]). A sequential effect algebra is a system  $(E; \oplus, \circ, 0, 1)$ , where  $(E; \oplus, 0, 1)$  is an effect algebra and  $\circ : E \times E \rightarrow E$  is a binary operation satisfying:

(SE1) The map  $b \mapsto a \circ b$  is additive for every  $a \in E$ , that is, if  $b \oplus c$  is defined, then  $a \circ b \oplus a \circ c$  is defined and  $a \circ (b \oplus c) = a \circ b \oplus a \circ c$ .

(SE2)  $1 \circ a = a$  for every  $a \in E$ .

(SE3) If  $a \circ b = 0$ , then  $a \circ b = b \circ a$ .

(SE4) If  $a \circ b = b \circ a$ , then  $a \circ b' = b' \circ a$  and  $a \circ (b \circ c) = (a \circ b) \circ c$  for every  $c \in E$ .

(SE5) If  $c \circ a = a \circ c$  and  $c \circ b = b \circ c$ , then  $c \circ (a \circ b) = (a \circ b) \circ c$  and  $c \circ (a \oplus b) = (a \oplus b) \circ c$ .

The operation  $\circ$  is called sequential product. This product provides a mechanism for describing quantum interference because if  $a \circ b \neq b \circ a$ , then  $a$  and  $b$  interfere ([7]).

Professor Gudder showed that for any two quantum effects  $B$  and  $C$ , the operation  $\circ$  defined by  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  satisfies conditions (SE1)-(SE5), and so is a sequential product of  $\mathcal{E}(\mathcal{H})$ . Thus,  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$  is a sequential effect algebra, we call it the sequential quantum effect algebra.

In 2005, Gudder presented 25 open problems in ([8]) to motive the study of sequential effect algebra theory, some of them are solved in recent years ([9, 10, 11, 12, 13, 14, 15]). In 2015, Wang etc. studied the entropies on sequential effect algebra ([16]).

In [6], Gudder gave five physically motivated conditions which fully characterize the sequential product on sequential quantum effect algebra  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$ , one of the conditions asked that the sequential product  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  is jointly continuous with respect to the strong operator topology. This showed that the continuity of sequential product operation  $\circ$  is an important and interesting problem, although the continuity of the operation  $\oplus$  and  $\ominus$  of effect algebras has been studied in [17, 18, 19, 20, 21], however, the continuity of the sequential product operation  $\circ$  of sequential effect algebras has not been considered until now.

In this paper, we will fill the gap for the sequential quantum effect algebra  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$ , that is, we will study the continuity of sequential product  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  on  $\mathcal{E}(\mathcal{H})$  with respect to the norm topology, weak operator topology, order convergence, order topology and interval topology. We will show that  $\circ$  on  $\mathcal{E}(\mathcal{H})$  is jointly continuous with respect to the norm topology,  $\circ$  is continuous in the second variable with respect to the weak operator topology, order convergence, order topology and interval topology. We will present examples to show that  $\circ$  is not continuous in the first variable with respect to the weak operator topology, order convergence, order topology and interval topology.

## 2. THE JOINTLY CONTINUITY OF SEQUENTIAL PRODUCT

**Definition 2.1.** . Let  $\mathcal{H}$  be a complex Hilbert Space. For any  $x \in \mathcal{H}$ , the equation  $P_x(T) = \|Tx\|$  defines a semi-norm  $P_x$  on  $\mathcal{B}(\mathcal{H})$ . The family of all semi-norms  $\{P_x : x \in \mathcal{H}\}$  gives rise to a topology on  $\mathcal{B}(\mathcal{H})$  called strong operator topology and denoted by  $SOT$ .

In the strong operator topology, an element  $T_0 \in \mathcal{B}(\mathcal{H})$  has a base of neighborhoods consisting of all sets of type

$$V(T_0 : x_1, \dots, x_m; \varepsilon) = \{T \in \mathcal{B}(\mathcal{H}) : \|(T - T_0)x_j\| < \varepsilon, j = 1, \dots, m\},$$

where  $\varepsilon$  is a positive number and  $x_1, \dots, x_m \in \mathcal{H}$ .

It can be proved  $T_\alpha \xrightarrow{SOT} T \Leftrightarrow \forall x \in \mathcal{H}, \|(T_\alpha - T)x\| \rightarrow 0$ .

Gudder had pointed out that  $\circ$  is jointly continuous in the strong operator topology([6]).

Next, we prove  $\circ$  is continuous with respect to the norm topology.

**Lemma 2.2.** ([22]). Let  $\{A_\alpha\}_{\alpha \in \Lambda}$  be a net in  $\mathcal{B}(\mathcal{H})$  and  $A \in \mathcal{B}(\mathcal{H})$ ,  $A_\alpha \geq 0, A \geq 0$ .

(1) If  $\|A_\alpha - A\| \rightarrow 0$ , then  $\|A_\alpha^{1/2} - A^{1/2}\| \rightarrow 0$ .

(2) If  $A_\alpha \xrightarrow{SOT} A$ , then  $A_\alpha^{1/2} \xrightarrow{SOT} A^{1/2}$ .

**Theorem 2.3.** The sequential product  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  on sequential quantum effect algebra  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$  is jointly continuous with respect to the norm topology.

That is, if  $A_\alpha \xrightarrow{\|\cdot\|} A$  and  $B_\alpha \xrightarrow{\|\cdot\|} B$ , then  $A_\alpha \circ B_\alpha \xrightarrow{\|\cdot\|} A \circ B$ .

$(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$ , one of the conditions asked that the sequential product  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$

*Proof.* By Lemma 2.2, we have  $A_\alpha^{1/2} \xrightarrow{\|\cdot\|} A^{1/2}$ . Then

$$\begin{aligned} \|A_\alpha \circ B_\alpha - A \circ B\| &= \|A_\alpha^{1/2}B_\alpha A_\alpha^{1/2} - A^{1/2}BA^{1/2}\| \\ &\leq \|A_\alpha^{1/2}B_\alpha A_\alpha^{1/2} - A_\alpha^{1/2}B_\alpha A^{1/2} + A_\alpha^{1/2}B_\alpha A^{1/2} - A_\alpha^{1/2}BA^{1/2} + A_\alpha^{1/2}BA^{1/2} - A^{1/2}BA^{1/2}\| \\ &\leq \|A_\alpha^{1/2}B_\alpha\| \|A_\alpha^{1/2} - A^{1/2}\| + \|A_\alpha^{1/2}\| \|B_\alpha - B\| \|A^{1/2}\| + \|A_\alpha^{1/2} - A^{1/2}\| \|BA^{1/2}\|. \end{aligned}$$

As  $\|A_\alpha^{1/2}B_\alpha\| \leq 1$ ,  $\|A_\alpha^{1/2}\| \|A^{1/2}\| \leq 1$  and  $\|BA^{1/2}\| \leq 1$ , we have

$$\|A_\alpha \circ B_\alpha - A \circ B\| \leq \|A_\alpha^{1/2} - A^{1/2}\| + \|B_\alpha - B\| + \|A_\alpha^{1/2} - A^{1/2}\| \rightarrow 0.$$

That is  $A_\alpha \circ B_\alpha \xrightarrow{\|\cdot\|} A \circ B$ . □

## 3. THE CONTINUITY OF THE SEQUENTIAL PRODUCT IN THE SECOND VARIABLE

**Definition 3.1.** ([22]). Suppose that  $\mathcal{V}$  is a linear space with scalar field  $K$ , and  $\mathcal{F}$  is a family of linear functionals on  $\mathcal{V}$ , which separates the points of  $\mathcal{V}$ . For any  $\rho \in \mathcal{F}$ , the equation  $P_\rho(x) = |\rho(x)|$  defines a semi-norm  $P_\rho$  on  $\mathcal{V}$ . The topology generated by  $\{P_\rho | \rho \in \mathcal{F}\}$  is called weak topology induced by  $\mathcal{F}$ .

**Definition 3.2.** ([22]). The weak operator topology on  $\mathcal{B}(\mathcal{H})$  is the weak topology induced by the family  $\mathcal{F}_w$  of linear functionals  $\omega_{x,y} : \mathcal{B}(\mathcal{H}) \rightarrow \mathbb{C}$  defined by the equation  $\omega_{x,y}(A) = \langle Ax, y \rangle$ ,  $x, y \in \mathcal{H}$ . The weak operator topology is denoted by  $WOT$ .

The family of sets of the form

$$V(T_0 : \omega_{x_1, y_1}, \dots, \omega_{x_m, y_m}; \varepsilon) = \{T \in \mathcal{B}(\mathcal{H}) : |\langle (T - T_0)x_j, y_j \rangle| < \varepsilon, j = 1, \dots, m\},$$

where  $\varepsilon$  is positive number and  $x_1, \dots, x_m, y_1, \dots, y_m \in \mathcal{H}$  constitutes a base of neighborhoods of  $T_0$  in WOT.

It can be proved that  $T_\alpha \xrightarrow{WOT} T \Leftrightarrow \forall x, y \in \mathcal{H}, \langle T_\alpha x, y \rangle \rightarrow \langle Tx, y \rangle \Leftrightarrow \forall x \in \mathcal{H}, \langle T_\alpha x, x \rangle \rightarrow \langle Tx, x \rangle$ .

**Theorem 3.3.** *The sequential product  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  on sequential quantum effect algebra  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$  is continuous in the second variable with respect to the weak operator topology. That is, if  $B_\alpha \xrightarrow{WOT} B$ , then  $A \circ B_\alpha \xrightarrow{WOT} A \circ B$  for each  $A \in \mathcal{E}(\mathcal{H})$ .*

*Proof.* As  $B_\alpha \xrightarrow{WOT} B$ ,  $\langle B_\alpha x, x \rangle \rightarrow \langle Bx, x \rangle$  for each  $x \in \mathcal{H}$ . Then  $\langle A \circ B_\alpha x, x \rangle = \langle A^{1/2}B_\alpha A^{1/2}x, x \rangle = \langle B_\alpha A^{1/2}x, A^{1/2}x \rangle \rightarrow \langle B A^{1/2}x, A^{1/2}x \rangle = \langle A^{1/2}B A^{1/2}x, x \rangle = \langle A \circ Bx, x \rangle$  for each  $x \in \mathcal{H}$ . That is  $A \circ B_\alpha \xrightarrow{WOT} A \circ B$ .  $\square$

We give an example to show that the continuity of  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  is not correct in the first variable with respect to WOT.

**Example 3.4.** Let  $\mathcal{H}$  be the complex separable Hilbert space  $l^2$  and  $\{e_n\}_{n=1}^\infty$  be its orthonormal basis. For each  $n$ , define

$$P_n e_i = \begin{cases} \frac{1}{2}e_1 + \frac{1}{2}e_{n+1}, & i = 1, i = n+1, \\ 0, & \text{others.} \end{cases}$$

and

$$P_0 e_i = \begin{cases} \frac{1}{2}e_1, & i = 1, \\ 0, & \text{others.} \end{cases}$$

It is easy to show that  $P_n$  is an orthogonal projection operator for each  $n$ . That is  $P_n \xrightarrow{WOT} P_0$  is clear.

Let

$$B e_i = \begin{cases} \frac{1}{2}e_1 + \frac{1}{2}e_2, & i = 1, i = 2, \\ 0, & \text{others.} \end{cases}$$

Then  $B \in \mathcal{E}(\mathcal{H})$ . Since  $\{P_n\}$  are orthogonal projection operators,

$$\langle P_n \circ Bx, x \rangle = \langle P_n^{\frac{1}{2}} B P_n^{\frac{1}{2}} x, x \rangle = \langle B P_n x, P_n x \rangle \rightarrow \langle \frac{1}{4} P_0 x, x \rangle$$

for each  $x \in l^2$ . That is  $P_n \circ B \xrightarrow{WOT} \frac{1}{4} P_0$ . However,  $P_0 \circ B = \frac{1}{2} P_0$ . So  $P_n \circ B$  is not convergent to  $P_0 \circ B$  with respect to WOT.

Let  $(P, \leq)$  be a poset. If  $\{a_\alpha\}_{\alpha \in \Lambda}$  is a net of  $P$  and  $a_\alpha \leq a_\beta$  when  $\alpha, \beta \in \Lambda$  and  $\alpha \preceq \beta$ , then we write  $a_\alpha \uparrow$ . Moreover, if  $a$  is the supremum of  $\{a_\alpha\}_{\alpha \in \Lambda}$ , i.e.  $a = \vee \{a_\alpha : \alpha \in \Lambda\}$ , then we write  $a_\alpha \uparrow a$ . Similarly, we may write  $a_\alpha \downarrow$  and  $a_\alpha \downarrow a$ .

We say that a net  $\{a_\alpha\}_{\alpha \in \Lambda}$  of  $P$  is order convergent to  $a \in P$  if there exist two nets  $\{u_\alpha\}_{\alpha \in \Lambda}$  and  $\{v_\alpha\}_{\alpha \in \Lambda}$  of  $P$  such that  $a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a$ . We denote order convergence as  $a_\alpha \xrightarrow{o} a$ . It can be proved that  $a_\alpha \xrightarrow{o} a \Rightarrow a_\alpha \xrightarrow{SOT} a$  ([24]).

**Lemma 3.5.** ([22]). *If  $\{A_\alpha\}$  is a monotone increasing net of self-adjoint operators on a Hilbert space  $\mathcal{H}$  and  $A_\alpha \leq I$  for all  $\alpha$ , then  $\{A_\alpha\}$  is strong-operator convergent to a self-adjoint operator  $A$ , and  $A$  is the least upper bound of  $\{A_\alpha\}$ .*

**Theorem 3.6.** *The sequential product  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  on sequential quantum effect algebra  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$  is continuous in the second variable with respect to the order convergence. That is, if  $B_\alpha \xrightarrow{o} B$ , then  $A \circ B_\alpha \xrightarrow{o} A \circ B$ .*

*Proof.* Let  $B_\alpha \xrightarrow{o} B$ . Then there exist two nets  $\{C_\alpha\}, \{D_\alpha\}$  such that  $C_\alpha \uparrow B$  and  $D_\alpha \downarrow B$  satisfying  $C_\alpha \leq B_\alpha \leq D_\alpha$ . It follows that  $A^{\frac{1}{2}}C_\alpha A^{\frac{1}{2}} \leq A^{\frac{1}{2}}B_\alpha A^{\frac{1}{2}} \leq A^{\frac{1}{2}}D_\alpha A^{\frac{1}{2}}$ . That is  $A \circ C_\alpha \leq A \circ B_\alpha \leq A \circ D_\alpha$ . It is clear that  $A \circ C_\alpha \uparrow$  and  $A \circ D_\alpha \downarrow$ . Since the order convergence is stronger than SOT, we have  $C_\alpha \xrightarrow{SOT} B$  and  $D_\alpha \xrightarrow{SOT} B$ . From the fact that  $\circ$  is jointly continuous with respect to SOT, it follows that  $A \circ C_\alpha \xrightarrow{SOT} A \circ B$  and  $A \circ D_\alpha \xrightarrow{SOT} A \circ B$ . By Lemma 3.5,  $A \circ C_\alpha \uparrow A \circ B$  and  $A \circ D_\alpha \downarrow A \circ B$ . That is,

$$A \circ B \uparrow A \circ C_\alpha \leq A \circ B_\alpha \leq A \circ D_\alpha \downarrow A \circ B.$$

Therefore,  $A \circ B_\alpha \xrightarrow{o} A \circ B$ .  $\square$

However, the conclusion is not correct in the first variable. That is,

**Example 3.7.** Let  $A_n = I - \frac{1}{n} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ . Then  $A_n \uparrow I$  and

$$\begin{aligned} A_n^{1/2} &= \frac{1}{2} \begin{pmatrix} \sqrt{1 - \frac{2}{n}} + 1 & \sqrt{1 - \frac{2}{n}} - 1 \\ \sqrt{1 - \frac{2}{n}} - 1 & \sqrt{1 - \frac{2}{n}} + 1 \end{pmatrix}, \\ A_n \circ B &= A_n^{1/2} B A_n^{1/2} = \frac{1}{2} \begin{pmatrix} 1 - \frac{1}{n} + \sqrt{1 - \frac{2}{n}} & -\frac{1}{n} \\ -\frac{1}{n} & 1 - \frac{1}{n} - \sqrt{1 - \frac{2}{n}} \end{pmatrix}, \\ \langle A_n \circ Bx, x \rangle &= \frac{1}{2} [(1 - \frac{1}{n} + \sqrt{1 - \frac{2}{n}})x_1^2 + (1 - \frac{1}{n} - \sqrt{1 - \frac{2}{n}})x_2^2 - \frac{2}{n}x_1x_2], \\ \langle I \circ Bx, x \rangle &= \langle Bx, x \rangle = x_1^2. \end{aligned}$$

Suppose  $A_n \circ B \xrightarrow{o} I \circ B = B$ . Then there exists an increasing net  $\{C_n\} \subseteq \mathcal{E}(\mathcal{H})$  and a decreasing net  $\{D_n\} \subseteq \mathcal{E}(\mathcal{H})$  satisfying  $B \uparrow C_n \leq A_n \circ B \leq D_n \downarrow B$ .

Let  $C_n = \begin{pmatrix} a_n & b_n \\ b_n & c_n \end{pmatrix}$ . Then  $\langle C_n x, x \rangle \leq \langle Bx, x \rangle$  for each  $x$ . It follows that  $b_n = c_n = 0$ ,  $a_n \uparrow 1$  and  $C_n = \begin{pmatrix} a_n & 0 \\ 0 & 0 \end{pmatrix}$  where  $a_n \geq 0$  and  $a_n \uparrow 1$ .  $\langle C_n x, x \rangle = a_n x_1^2$ .

For each  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  with  $x_1 \neq 0$ ,

$$\begin{aligned} &\langle (C_n - A_n \circ B)x, x \rangle \\ &= [a_n - \frac{1}{2}(1 - \frac{1}{n} + \sqrt{1 - \frac{2}{n}})]x_1^2 - \frac{1}{2}(1 - \frac{1}{n} - \sqrt{1 - \frac{2}{n}})x_2^2 + \frac{1}{n}x_1x_2 \\ &= \frac{1}{2}x_1^2 [-(1 - \frac{1}{n} - \sqrt{1 - \frac{2}{n}})(\frac{x_2}{x_1})^2 + \frac{2}{n}(\frac{x_2}{x_1}) + 2a_n - (1 - \frac{1}{n} + \sqrt{1 - \frac{2}{n}})]. \end{aligned}$$

Let  $t = \frac{x_2}{x_1}$ . Consider the function

$$f(t) = -(1 - \frac{1}{n} - \sqrt{1 - \frac{2}{n}})t^2 + \frac{2}{n}t + 2a_n - (1 - \frac{1}{n} + \sqrt{1 - \frac{2}{n}}).$$

$\Delta = (\frac{2}{n})^2 + 4(1 - \frac{1}{n} - \sqrt{1 - \frac{2}{n}})[2a_n - (1 - \frac{1}{n} + \sqrt{1 - \frac{2}{n}})] = 8a_n(1 - \frac{1}{n} - \sqrt{1 - \frac{2}{n}}) > 0$ . So there exists a  $t$  such that  $f(t) > 0$ . Therefore, there exists an  $x$  such that  $\langle (C_n - A_n \circ B)x, x \rangle > 0$ . This contradicts  $C_n \leq A_n \circ B$ . Thus, we have  $\{A_n \circ B\}$  is not order convergence to  $I \circ B = B$ .

Let  $(P, \leq)$  be a poset. Denote  $\mathcal{F} = \{F \subseteq P : \text{if } \{a_\alpha\}_{\alpha \in \Lambda} \subseteq F \text{ is a net and } \{a_\alpha\}_{\alpha \in \Lambda} \text{ is order convergent to } a \in P, \text{ then } a \in F\}$ . It can be proved that the family  $\mathcal{F}$  of subsets of  $P$  defines a topology  $\tau_o$  on  $P$  such that  $\mathcal{F}$  consists of all closed sets of this topology. The topology  $\tau_o$  is called the order topology on  $P$  ([18]).

It can be proved that the order topology  $\tau_o$  of  $P$  is the finest topology on  $P$  such that for each net  $\{a_\alpha\}_{\alpha \in \Lambda}$  of  $P$ , if  $a_\alpha \xrightarrow{o} a$ , then  $a_\alpha \xrightarrow{\tau_o} a$ . But the converse is not necessarily true ([18]).

**Theorem 3.8.** *The sequential product  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  on sequential quantum effect algebra  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$  is continuous in the second variable with respect to the order topology. That is, if  $B_\alpha \xrightarrow{\tau_o} B$ , then  $A \circ B_\alpha \xrightarrow{\tau_o} A \circ B$  for each  $A \in \mathcal{E}(\mathcal{H})$ .*

*Proof.* Firstly, let  $f : \mathcal{E}(\mathcal{H}) \rightarrow \mathcal{E}(\mathcal{H})$  defined by  $f(B) = A \circ B = A^{1/2}BA^{1/2}$ ,  $F$  be a closed set with respect to the order topology  $\tau_o$ ,  $F_1 = f^{-1}(F) = \{B \in \mathcal{E}(\mathcal{H}) : A^{1/2}BA^{1/2} \in F\}$ . Next, we prove that  $F_1$  is a closed set with respect to the order topology  $\tau_o$ . Let  $\{B_\alpha\} \subseteq F_1$  and  $B_\alpha \xrightarrow{o} B$ . Then  $A^{1/2}B_\alpha A^{1/2} \xrightarrow{o} A^{1/2}BA^{1/2}$  since  $\circ$  is continuous in the second variable with respect to the order convergence. Note that order convergence is stronger than order topology, we have  $A^{1/2}B_\alpha A^{1/2} \xrightarrow{\tau_o} A^{1/2}BA^{1/2}$ . As  $\{A^{1/2}B_\alpha A^{1/2}\} \subseteq F$  and  $F$  is closed in  $\tau_o$ , we obtain  $A^{1/2}BA^{1/2} \in F$ . Thus  $B \in F_1$  and  $F_1$  is closed in  $\tau_o$ . Therefore  $f$  is continuous according to  $\tau_o$ . That is  $B_\alpha \xrightarrow{\tau_o} B$  implies that  $A \circ B_\alpha \xrightarrow{\tau_o} A \circ B$  for each  $A \in \mathcal{E}(\mathcal{H})$ .  $\square$

Now, we show also that the conclusion is not correct in the first variable.

**Example 3.9.** Let  $\{A_n\}$  and  $B$  be defined as the same in Example 3.7. Then  $A_n \uparrow I$  implies  $A_n \xrightarrow{\tau_o} I$ . Suppose  $f(A) = A \circ B$  and  $f$  is continuous with respect to  $\tau_o$ . It follows that  $A_n \circ B \xrightarrow{\tau_o} I \circ B = B$ . Denote  $F = \{A_n \circ B\}$ . If  $\{A_n \circ B\}$  is order convergent and  $A_n \circ B \xrightarrow{o} M$ , then  $\langle A_n \circ Bx, x \rangle \rightarrow \langle Mx, x \rangle$  for each  $x$  since the order convergence is stronger than WOT. As in Example 3.7,  $\langle A_n \circ Bx, x \rangle \rightarrow \langle Bx, x \rangle$ . It follows that  $M = B$  which is contradict with Example 3.7. Thus  $\{A_n \circ B\}$  is not order convergent and  $F = \{A_n \circ B\}$  is closed in  $\tau_o$  by the definition. Let  $F_1 = f^{-1}(F) = \{A \in \mathcal{E}(\mathcal{H}) : A \circ B \in F\}$ . Then  $F_1$  is closed with respect to  $\tau_o$  as we have supposed  $f$  is continuous. As  $\{A_n\} \subseteq F_1$  and  $A_n \xrightarrow{o} I$ , we have  $I \in F_1$ . This implies  $B \in F$ . This is a contradiction. So  $f$  is not continuous with respect to  $\tau_o$ .

By the interval topology of a poset  $P$ , we mean the topology which is defined by taking all closed intervals  $[a, b]$  as a sub-basis of closed sets of  $P$ . We denote by  $\tau_I$  the interval topology. It can be verified that each closed interval  $[a, b]$  of a poset  $P$  is a closed set with respect to the order topology of  $P$ , so the interval topology is weaker than the order topology ([21]).

**Lemma 3.10.** ([21]). *Let  $(P, \leq)$  be a poset and  $\{a_\alpha\}_{\alpha \in \Lambda}$  be a net in  $(P, \leq)$ . Then  $a_\alpha \xrightarrow{\tau_I} a$  iff for any subnet  $\{a_\gamma\}_{\gamma \in \Upsilon}$ ,  $a_\gamma \geq r$  for  $r \in P$  implies  $a \geq r$  and  $a_\gamma \leq r$  for  $r \in P$  implies  $a \leq r$ .*

**Theorem 3.11.** *The sequential product  $B \circ C = B^{\frac{1}{2}}CB^{\frac{1}{2}}$  on sequential quantum effect algebra  $(\mathcal{E}(\mathcal{H}), 0, I, \oplus, \circ)$  is continuous in the second variable with respect to the order topology. That is, if  $B_\alpha \xrightarrow{\tau_I} B$ , then  $A \circ B_\alpha \xrightarrow{\tau_I} A \circ B$  for each  $A \in \mathcal{E}(\mathcal{H})$ .*

*Proof.* Let  $\{B_\gamma\}$  be any subnet of  $\{B_\alpha\}$  and  $A \circ B_\gamma \geq C_1$  for  $A, C_1 \in \mathcal{E}(\mathcal{H})$ . That is  $A^{1/2}B_\gamma A^{1/2} \geq C_1$ . For any  $\lambda > 0$ ,  $(\lambda I + A)^{1/2}B_\gamma(\lambda I + A)^{1/2} \geq C_1$  and  $(\lambda I + A)^{1/2}$  is invertible. Then we obtain

$$B_\gamma \geq (\lambda I + A)^{-1/2}C_1(\lambda I + A)^{-1/2}$$

for each  $\gamma$ . As  $B_\alpha \xrightarrow{\tau_I} B$ , by Lemma 3.10, we have

$$B \geq (\lambda I + A)^{-1/2}C_1(\lambda I + A)^{-1/2}.$$

So

$$(\lambda I + A)^{1/2}B(\lambda I + A)^{1/2} \geq C_1.$$

Let  $\lambda \rightarrow 0$ , we obtain  $A^{1/2}BA^{1/2} \geq C_1$ . That is  $A \circ B \geq C_1$ .

Next, let  $A \circ B_\gamma \leq C_2$ . Namely,  $A^{1/2}B_\gamma A^{1/2} \leq C_2$ . Let  $\lambda > 0$ . It is easy to prove that  $(\lambda I + A)^{1/2} \leq \sqrt{\lambda}I + A^{1/2}$ . So

$$\begin{aligned} (\lambda I + A)^{1/2}B_\gamma(\lambda I + A)^{1/2} &\leq (\sqrt{\lambda}I + A^{1/2})B_\gamma(\sqrt{\lambda}I + A^{1/2}) \\ &= \lambda B_\gamma + \sqrt{\lambda}(A^{1/2}B_\gamma + B_\gamma A^{1/2}) + A^{1/2}B_\gamma A^{1/2} \end{aligned}$$

It is also easy to prove  $\sqrt{\lambda}(A^{1/2}B_\gamma + B_\gamma A^{1/2}) \leq 2\sqrt{\lambda}I$ . So

$$(\lambda I + A)^{1/2}B_\gamma(\lambda I + A)^{1/2} \leq (\lambda + 2\sqrt{\lambda})I + C_2.$$

Since  $(\lambda I + A)^{1/2}$  is invertible, it follows

$$B_\gamma \leq (\lambda I + A)^{-1/2}[(\lambda + 2\sqrt{\lambda})I + C_2](\lambda I + A)^{-1/2}.$$

As  $B_\alpha \xrightarrow{\tau_I} B$ ,

$$B \leq (\lambda I + A)^{-1/2}[(\lambda + 2\sqrt{\lambda})I + C_2](\lambda I + A)^{-1/2}$$

and

$$(\lambda I + A)^{1/2}B(\lambda I + A)^{1/2} \leq (\lambda + 2\sqrt{\lambda})I + C_2.$$

Let  $\lambda \rightarrow 0$ , we have  $A^{1/2}BA^{1/2} \leq C_2$ . That is  $A \circ B \leq C_2$ . From Lemma 3.10 we obtain  $A \circ B_\alpha \xrightarrow{\tau_I} A \circ B$ .  $\square$

However, the conclusion is not correct in the first variable, too.

**Lemma 3.12.** ([22]). *The set  $\mathcal{P}(\mathcal{H})$  of orthogonal projections on  $\mathcal{H}$  is weak-operator dense in the set  $\mathcal{B}(\mathcal{H})_1^+$  of positive operators in the unit ball of  $\mathcal{B}(\mathcal{H})$ .*

**Example 3.13.** For  $\frac{I}{2}$ , by Lemma 3.12, there exists a sequence of projections  $\{E_n\}$  such that  $E_n \xrightarrow{WOT} \frac{I}{2}$ . As WOT is stronger than  $\tau_I$ , it follows that  $E_n \xrightarrow{\tau_I} \frac{I}{2}$ . For some  $x_0$  with  $\|x_0\| = 1$ , denote  $\mathcal{V} = \{F \in \mathcal{B}(\mathcal{H}) : |\langle (\frac{I}{2} - F)x_0, x_0 \rangle| < \frac{1}{3}\}$ . Then  $\mathcal{V}$  is a neighborhood of  $\frac{I}{2}$  for WOT. Without lost generality, we suppose that  $E_n \in \mathcal{V}$  for each  $n$ . It follows that

$$\langle \frac{I}{2}x_0, x_0 \rangle - \langle E_n x_0, x_0 \rangle \leq |\langle (\frac{I}{2} - E_n)x_0, x_0 \rangle| < \frac{1}{3}.$$

That is  $\langle E_n x_0, x_0 \rangle \geq \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ . So  $\bigwedge E_n \neq 0$ . Denote  $B = \bigwedge E_n$ , then  $B$  is an orthogonal projection.  $E_n \circ B = E_n B E_n = B$ ,  $\frac{I}{2} \circ B = \frac{1}{2}B$ .  $E_n \circ B = B \geq \frac{1}{2}B$ . However,  $\frac{I}{2} \circ B = \frac{1}{2}B$ . By Lemma 3.10,  $\{E_n \circ B\}$  is not convergent to  $\frac{I}{2} \circ B$  with respect to  $\tau_I$ .

**Acknowledgement:** This project is supported by National Natural Science Foundation of China (11101108, 11171301, 11571307) and by the Doctoral Programs Foundation of the Ministry of Education of China (J20130061).

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THE CONTINUITY OF SEQUENTIAL PRODUCT OF SEQUENTIAL QUANTUM EFFECT ALGEBRAS **9**

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